25-feb Lecture 8 Zoom

Wednesday, February 25, 2026 9:17 AM

announcements:

- office hours will be held on Zoom today 11am-1pm ET (same Zoom link)
- next week: review Mon, exam Wed
- review material posted on Brightspace by end of day Sat
- lecture will be recorded and posted

Section 3.1

recall $n \geq 2$ (derivative ex. 1^{st} deriv $\Rightarrow n = 1$)

second order format $G(x, y, y', y'') = 0$

G is linear provided it treats y and its derivatives in a "linear" way

linear 2^{nd} order DE will have form:

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

coefficient functions on open I

- don't have to be linear themselves
- continuous

linear

ex. $e^x y'' + (\cos x)y' + (1+\sqrt{x})y = \arctan x$

appear linearly e.g., no y^2

ex. not linear: $y'' + 3(y')^2 + 4y^3 = 0$

ex not linear: $y'' = \underbrace{yy'}_{\text{product}}$

Non-Homogeneous vs Homogeneous

different meaning than for 1^{st} order DE

associated homogeneous eqn

$x^2y'' + 2xy' + 3y = \cos x \longrightarrow x^2y'' + 2xy' + 3y = 0$

non-homogeneous

non-homogeneous

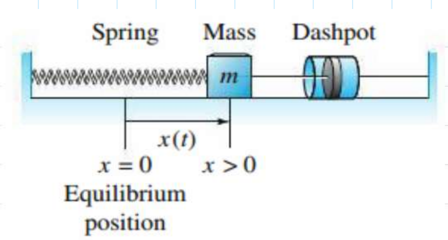
$F(x)$ represents an external influence to system

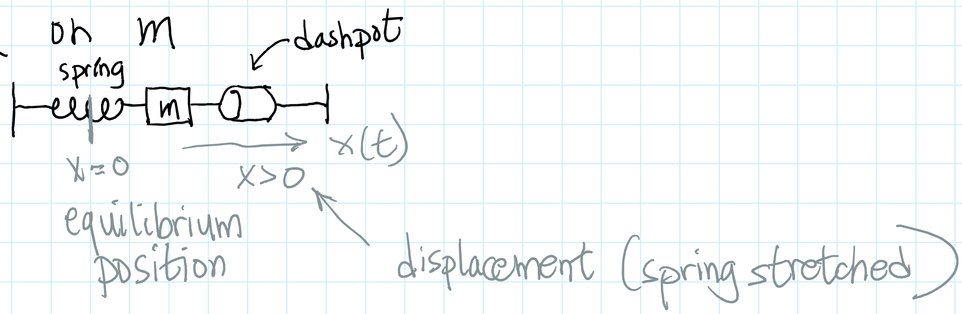
more formal definition : equation is homogeneous if $F(x)$ on RHS vanishes identically on some interval I
otherwise it's considered non-homogeneous

$F(x)$ is zero everywhere without exception

Application

mass, m, attached to a spring that exerts F_s on mass and there's a shock absorber (dashpot), which absorbs force F_R on m





$$F_s = -kx \qquad k>0$$

when spring is stretched : $x>0$ and $F_s<0$

when spring is compressed: $x<0$ and $F_s>0$

assuming F_R is also proportional, would be proportional to velocity s.t. $v = \frac{dx}{dt}$ of mass

$$F_R = -cv \qquad c>0$$
$$= -c\frac{dx}{dt}$$

$F_R<0$ if $v>0$ (motion to the right)

$F_R>0$ if $v<0$ (" left)

assume F_R, F_s are only forces present on m

assume F_R, F_s are only forces present on m
then can express acceleration simply : $a = \frac{dv}{dt}$

using $F = ma$
$= mx'' = F_s + F_R$

$m \frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$ move all to LHS

$m \frac{d^2x}{dt^2} + kx + c\frac{dx}{dt} = 0$ homogeneous

2nd order

" DE governs free vibrations of m "

if added force existed, $F(t)$, on m, would get added to RHS

becomes $mx'' = F_s + F_R + [\text{other } F(t)]$

$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$

non-homogeneous

" DE governs forced vibrations of m, influenced by external $F(t)$ "

consider $A(x)y'' + B(x)y' + C(x)y = F(x)$ where A, B, C, F are continuous on open I

assume $A(x) \neq 0$ on I

divide by A: $y'' + \frac{B}{A}y' + \frac{C}{A}y = \frac{F}{A}$

let $p(x) = \frac{B(x)}{A(x)}$ let $q(x) = \frac{C(x)}{A(x)}$ let $f(x) = \frac{F(x)}{A(x)}$

$y'' + p(x)y' + q(x)y = f(x)$

then its associated homogeneous, linear eqn is: $y'' + p(x)y' + q(x)y = 0$

useful property: given y_1 and y_2 are solutions to above homogeneous useful property: given y_1 and y_2 are solutions to above homogeneous linear eqn on I then the linear combination

$y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are constants

is also a solution on I

(nice proof of this in text)

ex. given $y'' + y = 0$ has sol'ns: $y_1(x) = \cos x$, $y_2(x) = \sin x$ verify this on your own

then $y(x) = 3\cos x - 2\sin x$ is also a sol'n

$\therefore$ a general sol'n of $y'' + y = 0$ is $y(x) = c_1 \cos x + c_2 \sin x$

c_1 is independent of c_2

IVP := initial value problem

IVP has unique sol'n for linear 2^{nd} order eqns

revisiting sol'n above:

$y(x) = 3\cos x - 2\sin x$ $\qquad$ $y'(x) = -3\sin x - 2\cos x$

$y(0) = 3(1) - 0$ $\qquad$ $y'(0) = -2 = c_2$

$y(0) = 3 = c_1$

initial values

conclusion: this is only (unique) sol'n given these initial values

generalize: sol'n $y(x) = b_0 \cos x + b_1 \sin x$ satisfies arbitrary initial condition $y(0) = b_0$ and $y'(0) = b_1$

ex. verify $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions of DE $y'' - 2y' + y = 0$

y_1: $e^x - 2e^x + e^x = 0$

$0 = 0$ $\square$

y_2: $y = xe^x$

$0 = 0$ ☐

y_2: $y = xe^x$

$y' = e^x + xe^x$

$y'' = e^x + e^x + xe^x = 2e^x + xe^x$

$\cancel{2e^x} + \cancel{xe^x} - 2\cancel{e^x} - \cancel{2xe^x} + \cancel{xe^x} \overset{?}{=} 0$

$0 = 0$ ☐

part 2 find a solution satisfying $y(0) = 3$, $y'(0) = 1$

write as linear combination: $y_3(x) = c_1 e^x + c_2 x e^x$ where c_1, c_2 constants

$y_3(0) = c_1 = 3$

$y_3'(x) = c_1 e^x + c_2(e^x + xe^x)$

$= c_1 e^x + c_2 e^x + c_2 \underline{x} e^x$

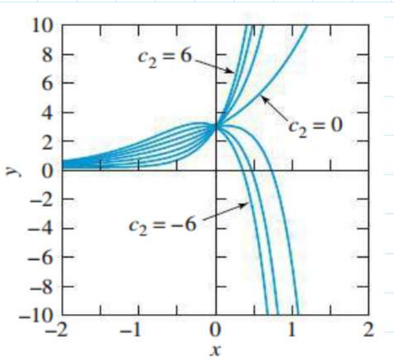
$y_3'(0) = c_1 + c_2 = 1$

$3 + c_2 = 1$

$c_2 = -2$

$\therefore$ sol'n to IVP is $y(x) = 3e^x - 2xe^x$

note: initial condition of $y(0) = 3$ alone does not yield unique sol'n



opens dialogue regarding linear independence:

two functions defined on open I are linearly independent on I if neither is a constant multiple of other

ex. linear dependent: $f = x$ $g = 2x$ $\frac{f}{g} = \frac{x}{2x} = \frac{1}{2}$ or $\frac{g}{f} = \frac{2x}{x}$

ex. linear dependent: $f = x \quad g = 2x \qquad \frac{f}{g}\ \frac{x}{2x} = \frac{1}{2}$ or $\frac{g}{f} = \frac{2x}{x} = 2$

constant functions

ex. " $f = \sin 2x, \quad g = \sin x \cos x$

$\sin 2x \to 2\sin x \cos x \qquad f = 2g$

ex. linear independent $\quad f = \sin x, \quad g = \cos x$

$\frac{f}{g} = \frac{\sin x}{\cos x} = \tan x \leftarrow$ not constant f

theorem: homogeneous eqn $y'' + p y' + q y = 0$ always has 2 (two) linearly independent sol'ns, y_1 and y_2, which can be expressed as $y = c_1 y_1 + c_2 y_2$ where c_1, c_2 are constants

To determine values of c_1 and c_2 use determinant

To find a 2x2 determinant of $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \qquad \det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

use $\begin{cases} y_1 = x_1 + 3x_2 \\ y_2 = 5x_1 + 7x_2 \end{cases} \xrightarrow[\text{composed of coefficients}]{\text{matrix is}} \begin{array}{cc} x_1 & x_2 \end{array} \begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix}$

$\det \begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 1(7) - 5(3) = -8$

in this context, use special determinant call Wronskian to establish linear independence:

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

ex. $W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x) = 1$

non-zero

non-zero

ex. $W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - e^x(xe^x)$

$x^a x^b = x^{a+b}$
$e^x e^x = e^{2x}$

$= e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}}$
$= e^{2x} \leftarrow$ will never be zero

ex. $f = kg$ where k is a constant

$$W(kg, g) = \begin{vmatrix} kg & g \\ kg' & g' \end{vmatrix} = kgg' - kgg' = 0$$

zero

theorem: suppose y_1, y_2 are two solutions of homogeneous 2^{nd} order linear DE in format $y'' + p(x)y' + q(x)y = 0$ on I

where p, q are cont on I

- if y_1, y_2 are linearly dep. then $W(y_1, y_2) = 0$ on I
- " linearly indep then " $\neq 0$ on each point of I

refine theorem: let y_1, y_2 be two linearly independent sol'ns of

$$y'' + p(x)y' + q(x)y = 0$$

if Y is a sol'n then exists c_1, c_2 s.t.

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) \text{ for all } x \text{ in } I$$

i.e., once found 2 linearly indep sol'ns of 2^{nd} order homogeneous linear eq'n (above)

then found all its sol'ns

call the linear combination $Y(x) = c_1 y_1 + c_2 y_2$ a general sol'n of this DE call the linear combination $Y(x) = c_1 y_1 + c_2 y_2$ a general sol'n of this DE